

SPECTRAL CHARACTERIZATION OF ABSOLUTELY REGULAR VECTOR-VALUED DISTRIBUTIONS.

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ABSTRACT. We study the reduced Beurling spectra $sp_{\mathcal{A},V}(F)$ of functions $F \in L^1_{loc}(\mathbb{J}, X)$ relative to certain function spaces $\mathcal{A} \subset L^\infty(\mathbb{J}, X)$ and $V \subset L^1(\mathbb{R})$, where \mathbb{J} is \mathbb{R}_+ or \mathbb{R} and X is a Banach space. We show that if F is bounded or slowly oscillating on \mathbb{J} with $0 \notin sp_{\mathcal{A},S}(F)$, where \mathcal{A} is $\{0\}$ or $C_0(\mathbb{J}, X)$ for example and $S = \mathcal{S}(\mathbb{R})$, then F is ergodic. This result is new even for $F \in BUC(\mathbb{J}, X)$ and $\mathcal{A} = C_0(\mathbb{J}, X)$. If F is ergodic and belongs to the space $S'_{ar}(\mathbb{J}, X)$ of absolutely regular distributions and if $sp_{C_0(\mathbb{J}, X), S}(F) = \emptyset$, then $\mathfrak{F} * \psi \in C_0(\mathbb{R}, X)$ for all $\psi \in \mathcal{S}(\mathbb{R})$. Here, $\mathfrak{F}|_{\mathbb{J}} = F$ and $\mathfrak{F}(\mathbb{R} \setminus \mathbb{J}) = 0$. We show that tauberian theorems for Laplace transforms follow from results about the reduced spectrum. Our results are more widely applicable than those of previous authors. We demonstrate this and the sharpness of our results through examples

§0. INTRODUCTION

The goal of this paper is to study the asymptotic behaviour of certain locally integrable functions $F : \mathbb{J} \rightarrow X$ where \mathbb{J} denotes \mathbb{R} or \mathbb{R}_+ and X is a complex Banach space. The study is motivated by tauberian theorems and their relevance to the behaviour of solutions of Cauchy problems in Banach spaces as in [2], [3], [4, Chapter 5], [6]. When $\mathbb{J} = \mathbb{R}_+$, the term “tauberian” has been used to describe theorems where the asymptotic behaviour of a function is deduced from properties of its Laplace transform, or equivalently its Laplace spectrum $sp^{\mathcal{L}}(F)$ (see [21], [4, p. 275]). Improvements were made by employing a smaller spectrum, the weak Laplace spectrum $sp^{w\mathcal{L}}(F)$ (see [4, p. 324], [16], [17]). In this paper we use an even smaller spectrum, the reduced (Beurling) spectrum $sp_{\mathcal{A}}(F)$ of F relative to various closed subspaces \mathcal{A} of $L^\infty(\mathbb{J}, X)$. This spectrum has been used before in this context (see [5], [6], [7], [16]). It has the advantage that it unifies the two cases $\mathbb{J} = \mathbb{R}_+$ and $\mathbb{J} = \mathbb{R}$. Moreover, we are able to consider functions whose Fourier

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transforms are not regular distributions and avoid some geometrical restrictions on X that were imposed in [17] for example. Importantly, spectral criteria for solutions of evolution equations are readily related to reduced spectra (see Theorem 3.5, [6]).

In section 1 we describe our notation and prove some preliminary results.

In section 2 we consider a more general spectrum $sp_{\mathcal{A},V}(F)$, the reduced spectrum of F relative to (\mathcal{A}, V) , where $V \subset L^1(\mathbb{R})$, a spectrum first studied in [9]. Typically, V is one of the spaces $\mathcal{D} = \mathcal{D}(\mathbb{R})$, $\mathcal{S} = \mathcal{S}(\mathbb{R})$ or $L^1(\mathbb{R})$. If $F \in L^\infty(\mathbb{R}, X)$, $\mathcal{A} = \{0\}$ and $V = L^1(\mathbb{R})$, then $sp_{\mathcal{A},V}(F) = sp^B(F)$ the classical Beurling spectrum. We study the conditions imposed on \mathcal{A} and relate them to previous ones in Proposition 2.1. Then we develop some basic properties of the reduced spectrum in Propositions 2.2 and 2.3. Our main results are stated in Theorems 2.5 and 2.6. The former deals with ergodicity. We show for example that if F is bounded or slowly oscillating on \mathbb{J} with $0 \notin sp_{\mathcal{A},\mathcal{S}}(F)$, where \mathcal{A} is $\{0\}$ or $C_0(\mathbb{J}, X)$, then F is ergodic. Theorem 2.6 deals with functions $F \in \mathcal{S}'_{ab}(\mathbb{J}, X)$, the space of absolutely regular distributions, with $sp_{C_0(\mathbb{J},X),\mathcal{S}}(F)$ countable. It is a generalized tauberian theorem providing spectral conditions under which F has various types of asymptotic behaviour. For example (Theorem 2.6 (iv)), if $sp_{C_0(\mathbb{J},X),\mathcal{S}}(F)$ is countable and non-empty and $\gamma_{-\omega}F$ is ergodic for each $\omega \in sp_{C_0(\mathbb{J},X),\mathcal{S}}(F)$, then $(\mathfrak{F} * \psi)|\mathbb{J}$ is asymptotically almost periodic for all $\psi \in \mathcal{S}(\mathbb{R})$. Versions of Theorem 2.5 and Theorem 2.6 (i), (ii), (iii), (v) are already known when $\mathbb{J} = \mathbb{R}_+$ and $sp_{\mathcal{A},\mathcal{S}}(F)$ is replaced by the larger spectrum $sp^{w\mathcal{L}}(F)$ (see [17]). Theorem 2.6 (iv), (vi) seem to be new for any spectrum. Proposition 2.7 states that if $F \in L^1_{loc}(\mathbb{J}, X)$ with $sp_{C_0(\mathbb{J},X),\mathcal{D}}(F) = \emptyset$ and if the convolution $(F * \psi)|\mathbb{J}$ is uniformly continuous for some $\psi \in \mathcal{D}(\mathbb{R})$ then $F * \psi \in C_0(\mathbb{R}, X)$. Chill [17, Proposition 2.1] obtained this same conclusion under the stronger assumptions that $F \in L^1_{loc}(\mathbb{R}, X)$ and $\widehat{F} \in \mathcal{S}'_{ar}(\mathbb{R}, X)$. In particular, if $F \in L^p(\mathbb{R}, X)$ where $1 \leq p < \infty$, then F satisfies the assumptions of Proposition 2.7. However, as is well-known, when $p > 2$ there are functions $F \in L^p(\mathbb{R}, X)$ for which \widehat{F} is not a regular distribution and so the result of [17] does not apply. Even when $1 \leq p \leq 2$ special geometry on X is required in order that every $F \in L^p(\mathbb{R}, X)$ has a Fourier transform which is regular.

In section 3 we establish some properties of weak Laplace, Laplace and Carleman spectra which are analogous to those of Beurling spectra. Also, if $F \in \mathcal{S}'_{ar}(\mathbb{R}_+, X)$ and $\mathcal{A} \supset C_0(\mathbb{R}_+, X)$ then $sp_{\mathcal{A},V}(F) \subset sp^{w\mathcal{L}}(F)$ (Proposition 3.2). As a conse-

quence, we strengthen several theorems about the asymptotic behaviour of absolutely regular tempered distributions, replacing Laplace and weak Laplace spectra by $sp_{C_0(\mathbb{R}_+, X), \mathcal{S}}(F)$ (see Remark 3.3). In Theorem 3.5 we obtain a spectral condition satisfied by bounded mild solutions of the evolution equation $\frac{du(t)}{dt} = Au(t) + \phi(t)$, $u(0) \in X$, $t \in \mathbb{J}$, where A is a closed linear operator on X and $\phi \in L^\infty(\mathbb{J}, X)$. This generalizes earlier results where it is assumed that $u, \phi \in BUC(\mathbb{J}, X)$ (see [4, Proposition 5.6.7], [6, Theorem 3.3, Corollary 3.4]).

§1. NOTATION, DEFINITIONS AND PRELIMINARIES

In this paper $\mathbb{R}_+ = [0, \infty)$, $\mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$, $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ and $\mathbb{C}_- = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$. By X we denote a complex Banach space. If Y, Z are locally convex topological spaces, $L(Y, Z)$ denotes the space of all bounded linear operators from Y to Z . The Schwartz spaces of test functions and rapidly decreasing functions are denoted by $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ respectively. Then $\mathcal{D}'(\mathbb{R}, X) = L(\mathcal{D}(\mathbb{R}), X)$ is the space of X -valued distributions and $\mathcal{S}'(\mathbb{R}, X) = L(\mathcal{S}(\mathbb{R}), X)$ is the space of X -valued tempered distributions (see [4, p. 482], [30, p. 149] for $X = \mathbb{C}$). The space of absolutely regular distributions is defined by

$$(1.1) \quad \mathcal{S}'_{ar}(\mathbb{J}, X) = \{H \in L^1_{loc}(\mathbb{J}, X) : H\varphi \in L^1(\mathbb{J}, X) \text{ for all } \varphi \in \mathcal{S}(\mathbb{R})\}.$$

The action of an element $S \in \mathcal{D}'(\mathbb{R}, X)$ or $\mathcal{S}'(\mathbb{R}, X)$ on $\varphi \in \mathcal{D}(\mathbb{R})$ or $\mathcal{S}(\mathbb{R})$ is denoted by $\langle S, \varphi \rangle$. If F is an X -valued function defined on \mathbb{J} and $s \in \mathbb{J}$ then $F_s, \Delta_s F, |F|$ stand for the functions defined on \mathbb{J} by $F_s(t) = F(t+s)$, $\Delta_s F(t) = F_s(t) - F(t)$ and $|F|(t) = \|F(t)\|$. Also $\|F\|_\infty = \sup_{t \in \mathbb{J}} \|F(t)\|$. If $F \in L^1_{loc}(\mathbb{J}, X)$ and $h > 0$, then $PF, M_h F$ and \check{F} (when $\mathbb{J} = \mathbb{R}_+$) denote the *indefinite integral*, *mollifier* and *reflection* of F defined respectively by $PF(t) = \int_0^t F(s) ds$, $M_h F(t) = (1/h) \int_0^h F(t+s) ds$ for $t \in \mathbb{J}$ and $\check{F}(t) = F(-t)$ for $t \in \mathbb{R}$. For $g \in L^1(\mathbb{R})$ and $F \in L^\infty(\mathbb{R}, X)$ or $g \in L^1(\mathbb{R}, X)$ and $F \in L^\infty(\mathbb{R})$ the *Fourier transform* \hat{g} and *convolution* $F * g$ are defined respectively by $\hat{g}(\omega) = \int_{-\infty}^\infty \gamma_{-\omega}(t) g(t) dt$ and $F * g(t) = \int_{-\infty}^\infty F(t-s)g(s) ds$, where $\gamma_\omega(t) = e^{i\omega t}$ for $\omega \in \mathbb{R}$. The *Fourier transform* of $S \in \mathcal{S}'(\mathbb{R}, X)$ is the tempered distribution \hat{S} defined by

$$(1.2) \quad \langle \hat{S}, \varphi \rangle = \langle S, \hat{\varphi} \rangle \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

Set $\hat{\mathcal{D}}(\mathbb{R}) = \{\hat{\varphi} : \varphi \in \mathcal{D}(\mathbb{R})\}$. The *Fourier transform* of $F \in L^1_{loc}(\mathbb{R}, X)$ is the distribution $\hat{F} \in L(\hat{\mathcal{D}}(\mathbb{R}), X)$ defined by

$$(1.3) \quad \langle \hat{F}, \psi \rangle = \langle F, \hat{\psi} \rangle \text{ for all } \psi \in \hat{\mathcal{D}}(\mathbb{R}).$$

Throughout the paper all integrals are Lebesgue-Bochner integrals ([4, pp. 6], [19, p. 318], [20, p. 76]). All convolutions are understood as convolutions of functions defined on \mathbb{R} . Given

$$(1.4) \quad F \in W(\mathbb{J}, X) \in \{L_{loc}^1(\mathbb{J}, X), \mathcal{S}'_{ar}(\mathbb{J}, X), L^\infty(\mathbb{J}, X)\},$$

we denote by

$$(1.5) \quad \mathfrak{F} \text{ the function given by } \mathfrak{F}|_{\mathbb{J}} = F \text{ and } \mathfrak{F}|_{(\mathbb{R} \setminus \mathbb{J})} = 0.$$

Then $\mathfrak{F} \in W(\mathbb{R}, X)$. In addition, if $g \in L_c^\infty(\mathbb{R}) = \{f \in L^\infty(\mathbb{R}) : f \text{ has compact support}\}$, then for some constant t_g

$$(1.6) \quad \mathfrak{F} * g \in W(\mathbb{R}, X) \cap C(\mathbb{R}, X) \text{ and if } \mathbb{J} = \mathbb{R}_+, \mathfrak{F} * g(t) = 0 \text{ for all } t \leq t_g.$$

It follows that if $h > 0$ and $s_h = (1/h)\chi_{(-h,0)}$, where $\chi_{(-h,0)}$ is the characteristic function of $(-h,0)$, then

$$(1.7) \quad \mathfrak{F} * s_h \in W(\mathbb{R}, X) \cap C(\mathbb{R}, X), \quad M_h F = (\mathfrak{F} * s_h)|_{\mathbb{J}} \text{ and} \\ \text{if } \mathbb{J} = \mathbb{R}_+, \mathfrak{F} * s_h(t) = 0 \text{ for all } t \leq -h.$$

We use convolutions of functions $F \in W = W(\mathbb{J}, X)$ and $g \in V = V(\mathbb{R}) \in \{\mathcal{D}(\mathbb{R}), \mathcal{S}(\mathbb{R}), L^1(\mathbb{R})\}$, with

$$(1.8) \quad V = \mathcal{D}(\mathbb{R}) \text{ if } W = L_{loc}^1(\mathbb{J}, X), V = \mathcal{S}(\mathbb{R}) \text{ if } W = \mathcal{S}'_{ar}(\mathbb{J}, X) \text{ and} \\ V = L^1(\mathbb{R}) \text{ if } W = L^\infty(\mathbb{J}, X).$$

The following properties of the convolution are repeatedly used (see [30, p. 156 (4)], [28, 7.19 Theorem (a), (b), pp. 179-180] when $X = \mathbb{C}$):

If $F \in W(\mathbb{J}, X)$ and $\varphi \in V(\mathbb{R})$ with W, V satisfying (1.8), then

$$(1.9) \quad \mathfrak{F} * \varphi \in W(\mathbb{R}, X) \cap C(\mathbb{R}, X).$$

Indeed, the cases $W = L_{loc}^1(\mathbb{J}, X)$ and $W = L^\infty(\mathbb{J}, X)$ are obvious. If $F \in \mathcal{S}'_{ar}(\mathbb{J}, X)$, then $|F| \in \mathcal{S}'_{ar}(\mathbb{J}, \mathbb{C})$. By [22, Theorem. (b)] there is an integer $k \in \mathbb{N}$ such that

$$(1.10.) \quad \frac{|F|}{w_k} = f \in L^1(\mathbb{J}), \text{ where } w_k(t) = (1 + t^2)^k.$$

Using (1.10), we easily conclude (1.9).

Moreover, if $\psi \in V(\mathbb{R})$ or $\psi \in L_c^\infty(\mathbb{R})$, then

$$(1.11) \quad (\mathfrak{F} * \varphi) * \psi = (\mathfrak{F} * \psi) * \varphi.$$

Also we need the following analogue of Wiener's theorem on Fourier series.

Lemma 1.1. *Let $f \in L^1(\mathbb{R})$ with $\widehat{f} \neq 0$ on a compact set K . Then there exists $g \in L^1(\mathbb{R})$ such that $\widehat{g} \cdot \widehat{f} = 1$ on K . Moreover, one can choose g such that \widehat{g} has compact support and, if $f \in \mathcal{S}(\mathbb{R})$, with $g \in \mathcal{S}(\mathbb{R})$.*

Proof. Choose a bounded open set U such that $K \subset U$ and $\widehat{f} \neq 0$ on \overline{U} the closure

of U . By [15, Proposition 1.1.5 (b), p. 22], there is $k \in L^1(\mathbb{R})$ such that $\widehat{k} \cdot \widehat{f} = 1$ on \overline{U} . Now, choose $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi = 1$ on K and $\text{supp } \varphi \subset \overline{U}$. By [28, Theorem 7.7 (b)] there is $\psi \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\psi} = \varphi$. Take $g = k * \psi$. Then \widehat{g} has compact support and if $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{g} \in \mathcal{D}(\mathbb{R})$ and so $g \in \mathcal{S}(\mathbb{R})$. \square

In the following proposition ψ will denote an element of $\mathcal{S}(\mathbb{R})$ with the properties:

$\widehat{\psi}$ has compact support, $\widehat{\psi}(0) = 1$ and ψ is non-negative.

An example of such ψ is given by $\psi = \widehat{\varphi}^2$, where $\varphi(t) = a e^{\frac{1}{t^2-1}}$ for $|t| \leq 1$, $\varphi = 0$ elsewhere on \mathbb{R} , with a some suitable constant.

Proposition 1.2. (i) *The sequence $\psi_n(t) = n\psi(nt)$ is an approximate identity for the space of uniformly continuous functions $UC(\mathbb{R}, X)$, that is $\lim_{n \rightarrow \infty} \|u * \psi_n - u\|_\infty = 0$ for all $u \in UC(\mathbb{R}, X)$.*

(ii) *$\lim_{h \searrow 0} \|M_h u - u\|_\infty = 0$ for all $u \in UC(\mathbb{J}, X)$. In particular if $M_h u \in BUC(\mathbb{J}, X)$ for all $h > 0$ then $u \in BUC(\mathbb{J}, X)$.*

Proof. (i) Given $u \in UC(\mathbb{R}, X)$ and $\varepsilon > 0$ there exists $k > 0$ such that $\|u(t+s) - u(t)\| \leq k|s| + \varepsilon$ for all $t, s \in \mathbb{R}$. In particular $u \in \mathcal{S}'_{ar}(\mathbb{R}, X)$. Also, $u * \psi_n(t) - u(t) = \int_{-\infty}^{\infty} [u(t - \frac{s}{n}) - u(t)] \psi(s) ds$ which gives $\|u * \psi_n - u\|_\infty \leq (k/n) \int_{-\infty}^{\infty} |s| \psi(s) ds + \varepsilon \int_{-\infty}^{\infty} \psi(s) ds$ and (i) follows.

(ii) Since $\|M_h u - u\|_\infty \leq \sup_{t \in \mathbb{J}, 0 \leq s \leq h} \|u(t+s) - u(t)\|$, part (ii) follows. \P

Proposition 1.3. (i) *Let $F \in W(\mathbb{R}_+, X)$ and $g \in V(\mathbb{R})$ with W, V satisfying (1.8). Then $\lim_{t \rightarrow -\infty} \|\mathfrak{F} * g(t)\| = 0$.*

(ii) *Let $F \in L^\infty(\mathbb{R}, X)$ and $F|_{\mathbb{R}_+} = 0$. Then $(F * f)|_{\mathbb{R}_+} \in C_0(\mathbb{R}_+, X)$ for each $f \in L^1(\mathbb{R})$.*

Proof. Part (ii) and the cases $F \in L^1_{loc}(\mathbb{R}_+, X)$ and $F \in L^\infty(\mathbb{R}_+, X)$ of part (i) can be shown by simple calculations. If $F \in \mathcal{S}'_{ar}(\mathbb{R}_+, X)$, then from (1.10) $|F|/w_k = f \in L^1(\mathbb{R}_+)$ for some $w_k(t) = (1+t^2)^k$. Since $\varphi \in \mathcal{S}(\mathbb{R})$, $\|w_k \varphi\|_\infty = c_k < \infty$. It follows that $\|\mathfrak{F} * \varphi(t)\| = \|\int_0^\infty \varphi(t-s) F(s) ds\| \leq \int_0^\infty |\varphi|(t-s) |F|(s) ds \leq c_k \int_0^\infty \frac{w_k(s)}{w_k(t-s)} f(s) ds$. Since $\frac{w_k(s)}{w_k(t-s)} \leq 1$ for each $t \leq 0, s \geq 0$ and $\lim_{t \rightarrow -\infty} \frac{w_k(s)}{w_k(t-s)} = 0$ for each $s \geq 0$, it follows that $\lim_{t \rightarrow -\infty} \|\mathfrak{F} * \varphi(t)\| = 0$ by the Lebesgue convergence theorem. \P

§2 REDUCED SPECTRA FOR REGULAR DISTRIBUTIONS

In this section we introduce the reduced spectrum $sp_{\mathcal{A}, V}(F)$ of a function $F \in$

$L^1_{loc}(\mathbb{J}, X)$ relative to \mathcal{A}, V , where $\mathcal{A} \subset L^\infty(\mathbb{J}, X)$ and $V \subset L^1(\mathbb{R})$. We usually impose the following conditions on \mathcal{A} .

- (2.1) \mathcal{A} is a closed subspace of $L^\infty(\mathbb{J}, X)$ and is *BUC-invariant*; that is
if $\phi \in BUC(\mathbb{R}, X)$ and $\phi|_{\mathbb{J}} \in \mathcal{A}$, then $\phi_a|_{\mathbb{J}} \in \mathcal{A}$ for each $a \in \mathbb{R}$.

The property of being *BUC-invariant* was first introduced in [5, (P.Λ), Definition 1.3.1] and called the Loomis property (P.Λ) for classes $\mathcal{A} \subset BUC(\mathbb{J}, X)$. The notion was extended to classes $\mathcal{A} \subset L^1_{loc}(\mathbb{J}, X)$ in [7, (1.III_{ub})]. In [9], this property was called *C_{ub}-invariance*.

We note that if $\mathbb{J} = \mathbb{R}$, then \mathcal{A} is *BUC-invariant* if and only if $\mathcal{A} \cap BUC(\mathbb{R}, X)$ is a translation invariant subspace of $BUC(\mathbb{R}, X)$. If $\mathbb{J} = \mathbb{R}_+$, then \mathcal{A} is *BUC-invariant* if and only if $\mathcal{A} \cap BUC(\mathbb{R}_+, X)$ is a positive invariant subspace of $BUC(\mathbb{R}_+, X)$ (that is $\phi_t \in \mathcal{A}$ for all $\phi \in \mathcal{A}$, $t \geq 0$) with the additional property that $u \in \mathcal{A}$ whenever $u \in BUC(\mathbb{R}_+, X)$ and $u_t \in \mathcal{A}$ for all $t \geq 0$. Such subspaces of $BUC(\mathbb{R}_+, X)$ were subsequently called *S*-biinvariant ([16, (1.1), p. 17], [3, §2]).

For \mathcal{A} satisfying (2.1), $V \subset L^1(\mathbb{R})$ and $F \in L^1_{loc}(\mathbb{J}, X)$, a point $\omega \in \mathbb{R}$ is called *(A, V)-regular* for F or \mathfrak{F} , if there is $\varphi \in V$ such that $\widehat{\varphi}(\omega) \neq 0$ and $(\mathfrak{F} * \varphi)|_{\mathbb{J}} \in \mathcal{A}$. The *reduced Beurling spectrum* of F or \mathfrak{F} relative to (\mathcal{A}, V) is defined by

$$(2.2) \quad sp_{\mathcal{A}, V}(F) = \{\omega \in \mathbb{R} : \omega \text{ is not an } (\mathcal{A}, V)\text{-regular point for } F\} = \\ \{\omega \in \mathbb{R} : \varphi \in V, (\mathfrak{F} * \varphi)|_{\mathbb{J}} \in \mathcal{A} \text{ implies } \widehat{\varphi}(\omega) = 0\} = sp_{\mathcal{A}, V}(\mathfrak{F}),$$

provided the convolution $\mathfrak{F} * \varphi$ and the restriction $(\mathfrak{F} * \varphi)|_{\mathbb{J}}$ are defined for all $\varphi \in V$ (see [10, (1.6)]). Further, if $H \in L^1_{loc}(\mathbb{R}, X)$ we also define (see [9, Definition 3.1])

$$(2.2^*) \quad sp_{\mathcal{A}, V}(H) = \{\omega \in \mathbb{R} : \varphi \in V, (H * \varphi)|_{\mathbb{J}} \in \mathcal{A} \text{ implies } \widehat{\varphi}(\omega) = 0\}.$$

It is clear that $sp_{\mathcal{A}, V}(F)$ and $sp_{\mathcal{A}, V}(H)$ are closed subsets of \mathbb{R} . If $\mathbb{J} = \mathbb{R}$, then $F = H|_{\mathbb{R}} = \mathfrak{F}$ and so (2.2) and (2.2*) give the same spectrum. If $\mathbb{J} = \mathbb{R}_+$ and $F = H|_{\mathbb{R}_+}$ we are interested in comparing $sp_{\mathcal{A}, V}(F)$ defined by (2.2) with $sp_{\mathcal{A}, V}(H)$ defined by (2.2*) (see Proposition 2.2).

For $F \in L^\infty(\mathbb{J}, X)$ and $V = L^1(\mathbb{R})$ we write $sp_{\mathcal{A}}(F) = sp_{\mathcal{A}, L^1(\mathbb{R})}(F)$.

If $F \in W(\mathbb{J}, X)$ and $V = V(\mathbb{R})$ with W, V satisfying (1.8), then the convolution $\mathfrak{F} * g$ and the restriction $(\mathfrak{F} * g)|_{\mathbb{J}}$ are defined for all $g \in V(\mathbb{R})$. So, $sp_{\mathcal{A}, V}(F)$ is well defined.

This is an extension of the definitions in [5, (4.1.1)], [6, (2.9)], [16, Definition 1.14, p. 24]. In those references the conditions on \mathcal{A} are more restrictive and $F \in L^\infty(\mathbb{R}, X)$. In particular, if $\mathcal{A} = \{0\}$ and $F \in L^\infty(\mathbb{R}, X)$ then $sp_0(F) =$

$sp_{\{0\}}(F)$ is the classical Beurling spectrum $sp^B(F)$ [27, p. 183]. If $F \in \mathcal{S}'_{ar}(\mathbb{R}, X)$, then $sp_{0,S}(F) = \text{supp } \widehat{F} = sp^C(F)$ (the Carleman spectrum). Indeed, the first equality is straightforward and the second is proved in [26, Proposition 0.5]. If $F \in L^\infty(\mathbb{J}, X)$, then $\mathfrak{F} * f \in BUC(\mathbb{R}, X)$ for all $f \in L^1(\mathbb{R})$. It follows that $sp_{\mathcal{A}}(F) = sp_{\mathcal{A} \cap BUC(\mathbb{J}, X)}(F)$.

Our approach of defining reduced spectra via convolutions is widely applicable. For $F \in BUC(\mathbb{J}, X)$ and $\mathcal{A} \subset BUC(\mathbb{J}, X)$, there is also an operator theoretical approach using C_0 -semigroups and groups. In [18] it is proved that the two approaches are equivalent for such F and \mathcal{A} (see also [9, Theorem 3.10]). In [24] there is an unsuccessful attempt to extend the operator theoretical approach to $F \in BC(\mathbb{J}, X)$ and $\mathcal{A} \subset BC(\mathbb{J}, X)$ (see [25]).

The space $\mathcal{A}_{\mathfrak{g}} = \mathfrak{g} \cdot AP(\mathbb{R}, X)$, where $\mathfrak{g}(t) = e^{it^2}$ for $t \in \mathbb{R}$, satisfies (2.1) and $\mathcal{A}_{\mathfrak{g}} \cap BUC(\mathbb{R}, X) = \{0\}$. We conclude that if $0 \neq F \in BC(\mathbb{R}, X)$, then $sp_{\mathcal{A}_{\mathfrak{g}}}(F) = sp^B(F) \neq \emptyset$. In particular, $sp_{\mathcal{A}_{\mathfrak{g}}}(F) \neq \emptyset$ for each $0 \neq F \in \mathcal{A}_{\mathfrak{g}}$. A sufficient condition to have the property $sp_{\mathcal{A}}(F) = \emptyset$ for each $F \in \mathcal{A} \subset L^\infty(\mathbb{J}, X)$ is the following inclusion

$$(2.3) \quad (\mathfrak{F} * f)|_{\mathbb{J}} \subset \mathcal{A} \text{ for each } F \in \mathcal{A} \text{ and } f \in L^1(\mathbb{R}).$$

Note that if $\mathcal{A} \subset BUC(\mathbb{J}, X)$ satisfies (2.1), then using the properties of Bochner integration we find \mathcal{A} satisfies (2.3). The space $\mathcal{A}_{\mathfrak{g}}$ does not satisfy (2.3).

Examples of spaces \mathcal{A} satisfying (2.1), (2.3) include (using $\mathcal{A}(\mathbb{J}, X) = \mathcal{A}(\mathbb{R}, X)|_{\mathbb{J}}$) $\{0\}, C_0(\mathbb{J}, X), AP(\mathbb{R}, X), LAP_b(\mathbb{R}, X), AA(\mathbb{R}, X), EAP(\mathbb{J}, X),$

$$AAP(\mathbb{J}, X) = AP(\mathbb{J}, X) \oplus C_0(\mathbb{J}, X), AAA(\mathbb{J}, X) = AA(\mathbb{J}, X) \oplus C_0(\mathbb{J}, X),$$

the spaces consisting respectively of the zero function (when $\mathbb{J} = \mathbb{R}$), continuous functions vanishing at infinity, almost periodic, Levitan bounded almost periodic, almost automorphic functions ([1], [5], [23]), Eberlein (weakly) almost periodic ([5, Definition 2.3.1]), asymptotically almost periodic functions (when $\mathbb{J} = \mathbb{R}_+$) ([5, Definitions 2.2.1, 2.3.1, (2.3.2)]) and asymptotically almost automorphic functions.

For $\lambda \in \mathbb{C}_+$ set

$$f_\lambda(t) = \begin{cases} e^{-\lambda t}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases} \text{ and } f_{-\lambda} = -\check{f}_\lambda.$$

Then $f_\lambda, \check{f}_\lambda \in L^1(\mathbb{R})$ for all $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. For $\phi \in L^\infty(\mathbb{R}, X)$ and $t \in \mathbb{R}$ define

$$\mathcal{C}\phi_t(\lambda) = \phi * \check{f}_\lambda(t) = \begin{cases} \int_0^\infty e^{-\lambda s} \phi(s+t) ds, & \text{if } \text{Re } \lambda > 0 \\ -\int_{-\infty}^0 e^{-\lambda s} \phi(s+t) ds, & \text{if } \text{Re } \lambda < 0 \end{cases}$$

Obviously, $\phi * \check{f}_\lambda \in BUC(\mathbb{R}, X)$ for all $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. We consider the property

$$(2.3^*) \quad (\mathfrak{F} * \check{f}_\lambda)|_{\mathbb{J}} \in \mathcal{A} \text{ for each } F \in \mathcal{A} \text{ and } \lambda \in \mathbb{C} \setminus i\mathbb{R}$$

as well as the following

$$(2.4) \quad H \in L^1_{loc}(\mathbb{R}, X) \text{ and } H|_{(-\infty, 0)} \text{ is bounded.}$$

Proposition 2.1. *Let $\mathcal{A} \subset L^\infty(\mathbb{J}, X)$ be a closed subspace.*

(i) *If \mathcal{A} is BUC-invariant and $\mathbb{J} = \mathbb{R}_+$, then $C_0(\mathbb{R}_+, X) \subset \mathcal{A}$. However, this is not necessarily true if $\mathbb{J} = \mathbb{R}$.*

(ii) *If \mathcal{A} satisfies (2.3), then \mathcal{A} is BUC-invariant.*

(iii) *\mathcal{A} satisfies (2.3*) if and only if \mathcal{A} satisfies (2.3).*

Proof. (i) By the BUC-invariance of \mathcal{A} , if $F \in BUC(\mathbb{R}, X)$ and F has compact support in $(-\infty, 0]$, then $F_t|_{\mathbb{R}_+} \in \mathcal{A}$ for all $t \in \mathbb{R}$. It follows that the space of continuous functions with compact support $C_c(\mathbb{R}_+, X) \subset \mathcal{A}$ and so $C_0(\mathbb{R}_+, X) \subset \mathcal{A}$ (see also the proof of Theorem 2.2.4 in [5, p. 13]). A counter-example for the case $\mathbb{J} = \mathbb{R}$ is $\mathcal{A} = AP(\mathbb{R}, X)$.

(ii) The case $\mathbb{J} = \mathbb{R}$: Since $L^1(\mathbb{R})$ is translation invariant, the set $\mathcal{A} * L^1(\mathbb{R})$ is translation invariant too. By Proposition 1.2 (i), $\mathcal{A} * L^1(\mathbb{R})$ is a dense subset of $\mathcal{A} \cap BUC(\mathbb{R}, X)$. As \mathcal{A} is closed, $\mathcal{A} \cap BUC(\mathbb{R}, X)$ is translation invariant.

The case $\mathbb{J} = \mathbb{R}_+$: By Proposition 2.2 (ii) below, we conclude that $(H_t * f)|_{\mathbb{R}_+} = (H * f_t)|_{\mathbb{R}_+} \in \mathcal{A}$ for each $H \in L^\infty(\mathbb{R}, X)$ such that $H|_{\mathbb{R}_+} \in \mathcal{A}$ and each $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$. If $H \in BUC(\mathbb{R}, X)$, then again using the approximate identity of Proposition 1.2 (i) we conclude that $H_t|_{\mathbb{R}_+} \in \mathcal{A}$ for each $t \in \mathbb{R}$. This gives (ii).

(iii) Obviously, (2.3) implies (2.3*). For the converse we begin by showing that $E = \text{span}\{f_\lambda : \text{Re } \lambda \neq 0\}$ is a dense subspace of $L^1(\mathbb{R})$. Indeed, if E is not dense in $L^1(\mathbb{R})$, then by the Hahn-Banach theorem there is $0 \neq \phi \in L^\infty(\mathbb{R}) = (L^1(\mathbb{R}))^*$ such that $\mathcal{C}\phi(\lambda) = \int_0^\infty e^{-\lambda t} \phi(t) dt = 0$ if $\text{Re } \lambda > 0$ and $\mathcal{C}\phi(\lambda) = -\int_0^\infty e^{\lambda t} \phi(-t) dt = 0$ if $\text{Re } \lambda < 0$. This means that the Carleman transform $\mathcal{C}\phi$ is zero on $\mathbb{C} \setminus i\mathbb{R}$ and implies $\text{sp}^C(\phi) = \emptyset$ and so $\phi = 0$ (see [26, Proposition 0.5 (ii)]). This is a contradiction showing that E is dense in $L^1(\mathbb{R})$. Given (2.3*) it follows that $(\mathfrak{F} * f)|_{\mathbb{J}} \in \mathcal{A}$ for each $F \in \mathcal{A}$ and $f \in E$. Since E is a dense subspace of $L^1(\mathbb{R})$ and \mathcal{A} is closed, (2.3) follows. ◀

Proposition 2.2. *Let $\mathcal{A} \subset L^\infty(\mathbb{J}, X)$ be a closed subspace satisfying (2.3). Assume that $H \in W(\mathbb{R}, X)$ satisfies (2.4) if $\mathbb{J} = \mathbb{R}_+$ and let $F = H|_{\mathbb{J}}$.*

(i) *If W, V satisfy (1.8), then $\text{sp}_{\mathcal{A}, V}(H) = \text{sp}_{\mathcal{A}, V}(F)$.*

(ii) If $H \in L^\infty(\mathbb{R}, X)$ and $H|_{\mathbb{J}} \in \mathcal{A}$, then $(H * f)|_{\mathbb{J}} \in \mathcal{A}$ for each $f \in L^1(\mathbb{R})$.

(iii) If $H \in L^\infty(\mathbb{R}, X)$, then $sp_{\mathcal{A}, \mathcal{S}}(F) = sp_{\mathcal{A}, \mathcal{S}}(H) = sp_{\mathcal{A}}(H) = sp_{\mathcal{A}}(F)$. In particular, $sp_0(H) = sp_{0, \mathcal{S}}(H) = sp^B(H)$.

(iv) If $W(\mathbb{R}, X) = \mathcal{S}'_{ar}(\mathbb{R}, X)$ and 0 is an (\mathcal{A}, V) -regular point for H , then there is $\delta > 0$ and $\psi \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\psi} \in \mathcal{D}(\mathbb{R})$, $\widehat{\psi} = 1$ on $[-\delta, \delta]$ and $(H * \psi)|_{\mathbb{J}} \in \mathcal{A}$.

Proof. (i) If $\mathbb{J} = \mathbb{R}$ there is nothing to prove so take $\mathbb{J} = \mathbb{R}_+$. For $\varphi \in V(\mathbb{R})$ we have $H * \varphi = \mathfrak{F} * \varphi + (H - \mathfrak{F}) * \varphi$. By Proposition 1.3(ii), $((H - \mathfrak{F}) * \varphi)|_{\mathbb{R}_+} \in C_0(\mathbb{R}_+, X)$, so by Proposition 2.1 (i) it follows that $(H * \varphi)|_{\mathbb{R}_+} \in \mathcal{A}$ if and only if $(\mathfrak{F} * \varphi)|_{\mathbb{R}_+} \in \mathcal{A}$.

(ii) Again we need only consider the case $\mathbb{J} = \mathbb{R}_+$. For $f \in L^1(\mathbb{R})$ we have $(H * f)|_{\mathbb{R}_+} = (\mathfrak{F} * f)|_{\mathbb{R}_+} + \xi$ where $\xi = ((H - \mathfrak{F}) * f)|_{\mathbb{R}_+}$. By (2.3), it follows that $(\mathfrak{F} * f)|_{\mathbb{R}_+} \in \mathcal{A}$ and by Propositions 1.3 (ii), 2.1 (i) we deduce that $\xi \in C_0(\mathbb{R}_+, X) \subset \mathcal{A}$. Hence $(H * f)|_{\mathbb{R}_+} \in \mathcal{A}$.

(iii) By part (i) we have $sp_{\mathcal{A}, \mathcal{S}}(H) = sp_{\mathcal{A}, \mathcal{S}}(F)$ and $sp_{\mathcal{A}, L^1(\mathbb{R})}(H) = sp_{\mathcal{A}, L^1(\mathbb{R})}(F)$. Moreover, it is clear that $sp_{\mathcal{A}, L^1(\mathbb{R})}(F) \subset sp_{\mathcal{A}, \mathcal{S}}(F)$. Conversely, we prove that a point $\omega_0 \in \mathbb{R}$ is $(\mathcal{A}, \mathcal{S})$ -regular for F if there is $h_0 \in L^1(\mathbb{R})$ such that $\widehat{h_0}(\omega_0) \neq 0$ and $(\mathfrak{F} * h_0)|_{\mathbb{J}} \in \mathcal{A}$. Choose $\delta > 0$ such that $\widehat{h_0} \neq 0$ on $[\omega_0 - \delta, \omega_0 + \delta]$ and by Lemma 1.1, $k_0 \in L^1(\mathbb{R})$ such that $\widehat{k_0} \cdot \widehat{h_0} = 1$ on $[\omega_0 - \delta, \omega_0 + \delta]$. Let $\varphi \in \mathcal{S}(\mathbb{R})$, $\widehat{\varphi}(\omega_0) \neq 0$ and $\text{supp } \widehat{\varphi} \subset [\omega_0 - \delta, \omega_0 + \delta]$. By (1.11) we have $\mathfrak{F} * \varphi = \mathfrak{F} * (h_0 * k_0 * \varphi) = (\mathfrak{F} * h_0) * (k_0 * \varphi)$. So, $(\mathfrak{F} * \varphi)|_{\mathbb{J}} \in \mathcal{A}$ by Proposition 1.3 (i) and part (ii). The second part follows by taking $\mathcal{A} = \{0\}$.

(iv) By (i), 0 is an $(\mathcal{A}, \mathcal{S})$ -regular point for F , so there is $\delta > 0$ and $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\varphi} \neq 0$ on $[-\delta, \delta]$ and $(\mathfrak{F} * \varphi)|_{\mathbb{J}} \in \mathcal{A}$. If $\mathbb{J} = \mathbb{R}$, then $H = F = \mathfrak{F}$ and so $(H * \varphi) \in \mathcal{A}$. If $\mathbb{J} = \mathbb{R}_+$, then $H * \varphi = \mathfrak{F} * \varphi + (H - \mathfrak{F}) * \varphi$. So $(H * \varphi)|_{\mathbb{R}_+} \in \mathcal{A}$ by Propositions 1.3(ii) and 2.1 (i). By Lemma 1.1, there is $g \in \mathcal{S}(\mathbb{R})$ such that $\widehat{g} \in \mathcal{D}(\mathbb{R})$ and $\widehat{\psi} = \widehat{\varphi * g} = 1$ on $[-\delta, \delta]$. Obviously $\psi \in \mathcal{S}(\mathbb{R})$ and $\widehat{\psi} \in \mathcal{D}(\mathbb{R})$. By (1.11) we have $H * \psi = (H * \varphi) * g$ and so $(H * \psi)|_{\mathbb{J}} \in \mathcal{A}$ by part (ii). \P

Proposition 2.3. *Let $\mathcal{A} \subset L^\infty(\mathbb{J}, X)$ be a closed subspace satisfying (2.3).*

(i) *Let W, V satisfy (1.8). If $F \in W(\mathbb{J}, X)$ and $g \in V(\mathbb{R})$ or $g \in L^\infty_c(\mathbb{R})$, then*

$$(2.5) \quad sp_{\mathcal{A}, V}(\mathfrak{F} * g) \subset sp_{\mathcal{A}, V}(F) \cap \text{supp } \widehat{g} \text{ and } sp_{\mathcal{A}, V}(F) = \cup_{h>0} sp_{\mathcal{A}, V}(M_h F).$$

(ii) *If $F \in \mathcal{S}'_{ar}(\mathbb{J}, X)$, $t \in \mathbb{R}$ and $0 \neq c \in \mathbb{C}$, then $sp_{\mathcal{A}, \mathcal{S}}(c\mathfrak{F}_t) = sp_{\mathcal{A}, \mathcal{S}}(F)$.*

(iii) *If $F, H \in \mathcal{S}'_{ar}(\mathbb{J}, X)$, then $sp_{\mathcal{A}, \mathcal{S}}(F + H) \subset sp_{\mathcal{A}, \mathcal{S}}(F) \cup sp_{\mathcal{A}, \mathcal{S}}(H)$.*

(iv) *If $F \in \mathcal{S}'_{ar}(\mathbb{J}, X)$ and $\gamma_\lambda \mathcal{A} \subset \mathcal{A}$ for all $\lambda \in \mathbb{R}$, then $sp_{\mathcal{A}, \mathcal{S}}(\gamma_\lambda F) = \lambda +$*

$sp_{\mathcal{A},\mathcal{S}}(F)$.

Proof. (i) Assume $\omega \notin sp_{\mathcal{A},V}(F)$. Then there is $\varphi \in V(\mathbb{R})$ with $\widehat{\varphi}(\omega) \neq 0$ and $(\mathfrak{F} * \varphi)|_{\mathbb{J}} \in \mathcal{A}$. By (1.9) and Proposition 1.3 (i), $\mathfrak{F} * \varphi \in BC(\mathbb{R}, X)$. By (1.11), we have $(\mathfrak{F} * g) * \varphi = (\mathfrak{F} * \varphi) * g = \mathfrak{F} * (\varphi * g)$. So, by Proposition 2.2 (ii), we get $((\mathfrak{F} * \varphi) * g)|_{\mathbb{J}} \in \mathcal{A}$ proving $\omega \notin sp_{\mathcal{A},V}(\mathfrak{F} * g)$. On the other hand if $\omega \notin \text{supp } \widehat{g}$, then there is $\varphi \in V(\mathbb{R})$ with $\widehat{\varphi}(\omega) \neq 0$ and $\varphi * g = 0$. So, $\omega \notin sp_{\mathcal{A},V}(\mathfrak{F} * g)$. For the case $\mathcal{A} = \{0\}$ see also [26, Proposition 0.6 (i)].

To prove the second part of (2.5) we note that $M_h F = (\mathfrak{F} * s_h)|_{\mathbb{J}}$, $\mathfrak{F} * s_h$ satisfies (2.4) if $\mathbb{J} = \mathbb{R}_+$ (see (1.7)) and $g = s_h \in L_c^\infty(\mathbb{R})$ for each $h > 0$. Hence $sp_{\mathcal{A},V}(\mathfrak{F} * s_h) \subset sp_{\mathcal{A},V}(F)$. By Proposition 2.2 (i), we have $sp_{\mathcal{A},V}(M_h F) = sp_{\mathcal{A},V}(F * s_h)$. It follows that $\cup_{h>0} sp_{\mathcal{A},V}(M_h F) \subset sp_{\mathcal{A},V}(F)$. Now, let $\omega \in sp_{\mathcal{A},V}(F)$. There is $h > 0$ such that $\widehat{s_h}(\omega) \neq 0$. Assume that $\omega \notin sp_{\mathcal{A},V}(M_h F) = sp_{\mathcal{A},V}(\mathfrak{F} * s_h)$. By Proposition 2.2 (iv), there is $\psi \in V(\mathbb{R})$ such that $\widehat{\psi}(\omega) \neq 0$ and $((\mathfrak{F} * s_h) * \psi)|_{\mathbb{J}} \in \mathcal{A}$. By (1.11), $(\mathfrak{F} * s_h) * \psi = \mathfrak{F} * (s_h * \psi)$. It follows that $(\mathfrak{F} * (s_h * \psi))|_{\mathbb{J}} \in \mathcal{A}$. Since $s_h * \psi \in V(\mathbb{R})$ and $\widehat{s_h * \psi}(\omega) \neq 0$, we conclude that $\omega \notin sp_{\mathcal{A},V}(F)$, a contradiction which shows $\omega \in sp_{\mathcal{A},V}(M_h F)$. This proves $sp_{\mathcal{A},V}(F) \subset \cup_{h>0} sp_{\mathcal{A},V}(M_h F)$.

The proofs of (ii), (iii), (iv) are similar to the case $\mathcal{A} = \{0\}$ ([26, Proposition 0.4]). \blacksquare

We recall (see [7, p. 118], [8, p. 1007], [12], [29]) that a function $F \in L_{loc}^1(\mathbb{J}, X)$ is called *ergodic* if there is a constant $m(F) \in X$ such that

$$\sup_{t \in \mathbb{J}} \left\| \frac{1}{T} \int_0^T F(t+s) ds - m(F) \right\| \rightarrow 0 \text{ as } T \rightarrow \infty.$$

The limit $m(F)$ is called the *mean* of F . The set of all such ergodic functions will be denoted by $\mathcal{E}(\mathbb{J}, X)$. We set $\mathcal{E}_0(\mathbb{J}, X) = \{F \in \mathcal{E}(\mathbb{J}, X) : m(F) = 0\}$, $\mathcal{E}_b(\mathbb{J}, X) = \mathcal{E}(\mathbb{J}, X) \cap L^\infty(\mathbb{J}, X)$, $\mathcal{E}_{b,0}(\mathbb{J}, X) = \{F \in \mathcal{E}_b(\mathbb{J}, X) : m(F) = 0\}$, $\mathcal{E}_{ub}(\mathbb{J}, X) = \mathcal{E}(\mathbb{J}, X) \cap BUC(\mathbb{J}, X)$ and $\mathcal{E}_{u,0}(\mathbb{J}, X) = \mathcal{E}_{ub}(\mathbb{J}, X) \cap \mathcal{E}_{b,0}(\mathbb{J}, X)$.

If $F \in L_{loc}^1(\mathbb{J}, X)$ and $\gamma_\omega F \in \mathcal{E}(\mathbb{J}, X)$ for some $\omega \in \mathbb{R}$, then

$$(2.6) \quad \gamma_\omega M_h F \in \mathcal{E}(\mathbb{J}, X) \text{ and } M_h \gamma_\omega F \in \mathcal{E}_b(\mathbb{J}, X) \text{ for all } h > 0.$$

Moreover, if $F \in L^\infty(\mathbb{J}, X)$ and $\gamma_\omega F \in \mathcal{E}_b(\mathbb{J}, X)$ for some $\omega \in \mathbb{R}$, then

$$(2.7) \quad \gamma_\omega (\mathfrak{F} * g)|_{\mathbb{J}} \in \mathcal{E}_{ub}(\mathbb{J}, X) \text{ for all } g \in L^1(\mathbb{R}).$$

To prove (2.6), note that

$$M_T \gamma_\omega M_h F = \gamma_\omega M_h \gamma_{-\omega} M_T \gamma_\omega F \text{ and } M_T M_h \gamma_\omega F = M_h M_T \gamma_\omega F.$$

It follows that $\gamma_\omega M_h F$, $M_h \gamma_\omega F \in \mathcal{E}(\mathbb{J}, X)$ for all $h > 0$. By [8, (2.4)], $M_h \gamma_\omega F \in$

$C_b(\mathbb{J}, X)$ and so $M_h \gamma_\omega F \in \mathcal{E}_b(\mathbb{J}, X)$. For (2.7) note that if $F \in L^\infty(\mathbb{J}, X)$, then $M_h F = (\mathfrak{F} * s_h)|_{\mathbb{J}}$ (see (1.7)) is bounded and uniformly continuous. So, $\gamma_\omega M_h F \in \mathcal{E}_{ub}(\mathbb{J}, X)$ by (2.6). It follows that $\gamma_\omega(\mathfrak{F} * g)|_{\mathbb{J}} \in \mathcal{E}_{ub}(\mathbb{J}, X)$ for any step function g . Since step functions are dense in $L^1(\mathbb{R})$, (2.7) follows.

Also, we note that

$$(2.8) \quad \mathcal{E}_u(\mathbb{J}, X) := UC(\mathbb{J}, X) \cap \mathcal{E}(\mathbb{J}, X) = \mathcal{E}_{ub}(\mathbb{J}, X).$$

This follows by Proposition 1.2 (ii) using (2.6) (see also [8, Proposition 2.9]).

Next we recall the definition of the class of slowly oscillating functions

$$SO(\mathbb{J}, X) = UC(\mathbb{J}, X) + L_{loc,0}^1(\mathbb{J}, X),$$

where (see [17, Lemma 1.6], [4, Proposition 4.2.2] for the case $\mathbb{J} = \mathbb{R}_+$)

$$L_{loc,0}^1(\mathbb{J}, X) = \{F \in L_{loc}^1(\mathbb{J}, X) : \lim_{|t| \rightarrow \infty, t \in \mathbb{J}} F(t) = 0\}$$

It follows that if $F \in L_{loc,0}^1(\mathbb{J}, X)$ and $\psi \in \mathcal{S}(\mathbb{R})$, then

$$(2.9) \quad F \in \mathcal{E}_0(\mathbb{J}, X), \mathfrak{F} \in L_{loc,0}^1(\mathbb{R}, X),$$

$$(2.10) \quad M_h F \in C_0(\mathbb{J}, X) \text{ for all } h > 0 \text{ and } (\mathfrak{F} * \psi) \in C_0(\mathbb{R}, X).$$

Lemma 2.4. *If $F \in L^\infty(\mathbb{R}, X)$ and $0 \notin sp_{0,S}(F)$, then $F \in \mathcal{E}_{b,0}(\mathbb{R}, X)$. If $F \in SO(\mathbb{R}, X)$ and $0 \notin sp_{0,S}(F)$, then $F \in \mathcal{E}_{u,0}(\mathbb{R}, X) + L_{loc,0}^1(\mathbb{R}, X) \subset \mathcal{E}_0(\mathbb{R}, X)$.*

Proof. If $F \in L^\infty(\mathbb{R}, X)$ and $0 \notin sp_{0,S}(F)$, then by Proposition 2.2 (iii) $0 \notin sp^B(F)$. By [11, Corollary 4.4], $PF \in BUC(\mathbb{R}, X)$, and hence $F \in \mathcal{E}_{b,0}(\mathbb{R}, X)$. If $F \in SO(\mathbb{R}, X)$, let $F = \Phi + \xi$ with $\Phi \in UC(\mathbb{R}, X)$ and $\xi \in L_{loc,0}^1(\mathbb{R}, X)$. By (2.5), we have $sp_{0,S}(M_h F) \subset sp_{0,S}(F)$ and so $0 \notin sp_{0,S}(M_h F)$ for all $h > 0$. Since $M_h F \in UC(\mathbb{R}, X)$, we get $M_h F \in BUC(\mathbb{R}, X)$ for all $h > 0$ by [11, Theorem 4.2]. Since $M_h \xi \in C_0(\mathbb{R}, X)$ by (2.10), we get $M_h \Phi \in BUC(\mathbb{R}, X)$ for all $h > 0$. This implies $\Phi = \lim_{h \rightarrow 0} M_h \Phi \in BUC(\mathbb{R}, X)$ by Proposition 1.2 (ii). Choose $\delta > 0$ and $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\varphi} = 1$ on $[-\delta, \delta]$. By (2.9), (2.10) it follows that $\eta = (\xi - \xi * \varphi) \in \mathcal{E}_0(\mathbb{R}, X)$. Set $\Psi = \Phi + \xi * \varphi = F - \eta$. Since $0 \notin sp_{0,S}(\eta) \cup sp_{0,S}(\Phi)$, we get $0 \notin sp_{0,S}(\Psi)$. Since by (2.10) $\Psi \in BUC(\mathbb{R}, X)$, we conclude that $P\Psi \in BUC(\mathbb{R}, X)$, by [11, Corollary 4.4]. This implies that $\Psi \in \mathcal{E}_{u,0}(\mathbb{R}, X)$ and proves that $F = \Psi + \eta \in \mathcal{E}_0(\mathbb{R}, X)$. \blacksquare

We are now ready to state and prove our main results.

Theorem 2.5. *Let $\mathcal{A} \subset L^\infty(\mathbb{J}, X)$ be a closed subspace satisfying (2.3) and $\gamma_\lambda \mathcal{A} \subset \tilde{\mathcal{E}} \in \{\mathcal{E}(\mathbb{J}, X), \mathcal{E}_0(\mathbb{J}, X)\}$ for all $\lambda \in \mathbb{R}$. Let $F \in \mathcal{S}'_{ar}(\mathbb{J}, X)$ and $\omega \notin sp_{\mathcal{A},S}(F)$.*

(i) If $F \in L^\infty(\mathbb{J}, X)$, then $\gamma_{-\omega}F \in \tilde{\mathcal{E}}$.

(ii) If $\gamma_{-\omega}F \in SO(\mathbb{J}, X)$, then $\gamma_{-\omega}F \in (\mathcal{E}_{ub}(\mathbb{J}, X) + L^1_{loc,0}(\mathbb{J}, X)) \cap \tilde{\mathcal{E}}$. If also $\mathcal{A} = C_0(\mathbb{J}, X)$, then $\gamma_{-\omega}F \in \mathcal{E}_0(\mathbb{J}, X)$ and if $\gamma_{-\omega}F \in UC(\mathbb{J}, X)$, then $\gamma_{-\omega}F \in \mathcal{E}_{u,0}(\mathbb{J}, X)$.

Proof. Replacing $\gamma_{-\omega}F$ by F , we may assume $\omega = 0$ and $0 \notin sp_{\mathcal{A},S}(F)$. So, by Proposition 2.2 (iii) there is $\delta > 0$ and $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\text{supp } \hat{\varphi}$ is compact, $\hat{\varphi} = 1$ in a neighbourhood of 0 and $(\mathfrak{F} * \varphi)|_{\mathbb{J}} \in \mathcal{A} \subset \tilde{\mathcal{E}}$. Set $G = \mathfrak{F} - \mathfrak{F} * \varphi$.

(i) If $F \in L^\infty(\mathbb{J}, X)$, then $G \in L^\infty(\mathbb{R}, X)$ and $0 \notin sp_0(G) = sp_{0,S}(G)$ by Proposition 2.2 (iii) (see also [9, (3.3), (3.11)]). By Lemma 2.4, we get $G \in \mathcal{E}_{b,0}(\mathbb{R}, X)$. It follows that $F = [G + F * \varphi]|_{\mathbb{J}} \in \tilde{\mathcal{E}}$.

(ii) If $F \in SO(\mathbb{J}, X)$, then by (2.10) $G \in SO(\mathbb{R}, X)$ and $0 \notin sp_{0,S}(G)$. By Lemma 2.4, it follows that $G \in \mathcal{E}_{u,0}(\mathbb{R}, X) + L^1_{loc,0}(\mathbb{R}, X) \subset \mathcal{E}_0(\mathbb{R}, X)$. This implies $F = [G + F * \varphi]|_{\mathbb{J}} \in \tilde{\mathcal{E}}$. Obviously if $\mathcal{A} = C_0(\mathbb{J}, X)$, then $F \in \mathcal{E}_0(\mathbb{J}, X)$ and if $F \in UC(\mathbb{J}, X)$, then $F \in \mathcal{E}_{u,0}(\mathbb{J}, X)$. \blacktriangleleft

Theorem 2.6. Assume that $F \in \mathcal{S}'_{ar}(\mathbb{J}, X)$, $sp_{C_0(\mathbb{J}, X), S}(F)$ is countable and $\gamma_{-\omega}F \in \mathcal{E}(\mathbb{J}, X)$ for all $\omega \in sp_{C_0(\mathbb{J}, X), S}(F)$.

(i) If $F \in UC(\mathbb{J}, X)$, then $F \in AAP(\mathbb{J}, X)$. If also $sp_{C_0(\mathbb{J}, X), S}(F) = \emptyset$, then $F \in C_0(\mathbb{J}, X)$.

(ii) If $F \in SO(\mathbb{J}, X)$, then $F \in AP(\mathbb{J}, X) \oplus L^1_{loc,0}(\mathbb{J}, X)$. If also $sp_{C_0(\mathbb{J}, X), S}(F) = \emptyset$, then $F \in L^1_{loc,0}(\mathbb{J}, X)$.

(iii) If $F = H|_{\mathbb{J}}$ where $H \in L^\infty(\mathbb{R}, X)$ and if $f \in L^1(\mathbb{R})$, then $(H * f)|_{\mathbb{J}} \in AAP(\mathbb{J}, X)$.

(iv) If $sp_{C_0(\mathbb{J}, X), S}(F) \neq \emptyset$ and if $\psi \in \mathcal{S}(\mathbb{R})$, then $(\mathfrak{F} * \psi)|_{\mathbb{J}} \in AAP(\mathbb{J}, X)$.

(v) If $sp_{C_0(\mathbb{J}, X), S}(F) = \emptyset$ and if $\psi \in \mathcal{S}(\mathbb{R})$ with $\hat{\psi} \in \mathcal{D}(\mathbb{R})$, then $\mathfrak{F} * \psi \in C_0(\mathbb{R}, X)$.

(vi) If $sp_{C_0(\mathbb{J}, X), S}(F) = \emptyset$ and either $F \in \mathcal{E}(\mathbb{J}, X)$ or more generally $M_h F \in BC(\mathbb{J}, X)$ for all $h > 0$ and if $\psi \in \mathcal{S}(\mathbb{R})$, then $\mathfrak{F} * \psi \in C_0(\mathbb{R}, X)$.

Proof. (i) First, we note that $\mathcal{A} = C_0(\mathbb{J}, X)$ satisfies the assumptions of Theorem 2.5 with $\mathcal{E} = \mathcal{E}_0(\mathbb{J}, X)$. If $0 \notin sp_{C_0(\mathbb{J}, X), S}(F)$, then by Theorem 2.5, $F \in \mathcal{E}_{u,0}(\mathbb{J}, X)$. If $0 \in sp_{C_0(\mathbb{J}, X), S}(F)$, then from $F \in UC(\mathbb{J}, X)$ and (2.8) we get $F \in \mathcal{E}_{ub}(\mathbb{J}, X)$. Let $\tilde{F} \in BUC(\mathbb{R}, X)$ be an extension of F . Since $C_0(\mathbb{J}, X) \subset AAP(\mathbb{J}, X)$ we get $sp_{AAP(\mathbb{J}, X)}(\tilde{F}) \subset sp_{C_0(\mathbb{J}, X)}(\tilde{F})$. It follows from Proposition 2.2 (iii) that $sp_{AAP(\mathbb{J}, X)}(\tilde{F})$ is countable. By [5, Theorem 4.2.6] $F = \tilde{F}|_{\mathbb{J}} \in AAP(\mathbb{J}, X)$. If $sp_{C_0(\mathbb{J}, X), S}(F) = \emptyset$, then $\gamma_\lambda F \in \mathcal{E}_{u,0}(\mathbb{J}, X)$ for all $\lambda \in \mathbb{R}$. This implies $F \in C_0(\mathbb{J}, X)$.

(ii) Let $F = u + \xi$, where $u \in UC(\mathbb{J}, X)$, $\xi \in L_{loc,0}^1(\mathbb{J}, X)$. We note that $sp_{C_0(\mathbb{J}, X), \mathcal{S}}(\xi) = \emptyset$ by (2.10) and $\gamma_\lambda \xi \in \mathcal{E}(\mathbb{J}, X)$ for all $\lambda \in \mathbb{R}$, by (2.9). Also, we have $sp_{C_0(\mathbb{J}, X), \mathcal{S}}(M_h F)$ is countable by (2.5). By (2.6), we get $\gamma_{-\omega} M_h F \in \mathcal{E}(\mathbb{J}, X)$ for all $\omega \in sp_{C_0(\mathbb{J}, X), \mathcal{S}}(F)$. It follows that $sp_{C_0(\mathbb{J}, X), \mathcal{S}}(M_h u)$ is countable and $\gamma_{-\omega} M_h u \in \mathcal{E}(\mathbb{J}, X)$ for all $\omega \in sp_{C_0(\mathbb{J}, X), \mathcal{S}}(F)$. So, by part (i), we conclude that $M_h u \in AAP(\mathbb{J}, X)$ for all $h > 0$. By Proposition 1.2 (ii), $u = \lim_{h \rightarrow 0} M_h u \in AAP(\mathbb{J}, X)$. It follows that $F \in AP(\mathbb{J}, X) \oplus L_{loc,0}^1(\mathbb{J}, X)$. If $sp_{C_0(\mathbb{J}, X), \mathcal{S}}(F) = \emptyset$, then $\gamma_\lambda F \in \mathcal{E}_{u,0}(\mathbb{J}, X)$ for all $\lambda \in \mathbb{R}$. This implies that $F \in L_{loc,0}^1(\mathbb{J}, X)$.

(iii) Let $f \in L^1(\mathbb{R})$. Then $\mathfrak{F} * f \in BUC(\mathbb{R}, X)$. By (2.5), we deduce that $sp_{C_0(\mathbb{J}, X), \mathcal{S}}(\mathfrak{F} * f)$ is countable. By (2.7) we find that $\gamma_{-\omega}(\mathfrak{F} * f)|\mathbb{J} \in \mathcal{E}_{ub}(\mathbb{J}, X)$ for all $\omega \in sp_{C_0(\mathbb{J}, X), \mathcal{S}}(\mathfrak{F} * f)$. It follows that $(\mathfrak{F} * f)|\mathbb{J} \in AAP(\mathbb{J}, X)$, by part (i). By Proposition 1.3 (i), we have $((H - \mathfrak{F}) * f)|\mathbb{J} \in C_0(\mathbb{J}, X)$. Hence $(H * f)|\mathbb{J} \in AAP(\mathbb{J}, X)$.

(iv) Without loss of generality we may assume $0 \in sp_{C_0(\mathbb{J}, X), \mathcal{S}}(F)$. Then $F \in \mathcal{E}(\mathbb{J}, X)$ and so by (2.6), $(\mathfrak{F} * s_h)|\mathbb{J} = M_h F \in \mathcal{E}_b(\mathbb{J}, X)$ and $\gamma_{-\omega} M_h F \in \mathcal{E}_b(\mathbb{J}, X)$ for all $h > 0$ and $\omega \in sp_{C_0(\mathbb{J}, X), \mathcal{S}}(F)$. By (2.5), $sp_{C_0(\mathbb{J}, X), \mathcal{S}}(M_h F) \subset sp_{C_0(\mathbb{J}, X), \mathcal{S}}(F)$. Therefore, by part (iii), $((\mathfrak{F} * s_h) * g)|\mathbb{J} \in AAP(\mathbb{J}, X)$ for all $g \in L^1(\mathbb{R})$. Take $\psi \in \mathcal{S}(\mathbb{R})$. It follows that $M_h(\mathfrak{F} * \psi)|\mathbb{J} = ((\mathfrak{F} * s_h) * \psi)|\mathbb{J} \in AAP(\mathbb{J}, X)$ and also $(\Delta_h(\mathfrak{F} * \psi))|\mathbb{J} = (\mathfrak{F} * \Delta_h \psi)|\mathbb{J} = (\mathfrak{F} * h M_h \psi')|\mathbb{J} = (\mathfrak{F} * (s_h * \psi'))|\mathbb{J} = ((\mathfrak{F} * s_h) * \psi')|\mathbb{J} \in AAP(\mathbb{J}, X)$. By [8, Proposition 1.4], one gets $(\mathfrak{F} * \psi)|\mathbb{J}$ is uniformly continuous. This implies $(\mathfrak{F} * \psi)|\mathbb{J} = \lim_{h \searrow 0} M_h(\mathfrak{F} * \psi)|\mathbb{J} \in AAP(\mathbb{J}, X)$, by Proposition 1.2 (ii).

(v) Let $\omega \in K = \text{supp } \widehat{\psi}$. Since $C_0(\mathbb{J}, X)$ satisfies (2.1), (2.3), by Proposition 2.2 (iv), there is $f^\omega \in \mathcal{S}(\mathbb{R})$ such that $\widehat{f^\omega}$ has compact support, $\widehat{f^\omega} = 1$ on an open neighbourhood $V(\omega)$ of ω and $(\mathfrak{F} * f^\omega)|\mathbb{J} \in C_0(\mathbb{J}, X)$. Take $k^\omega = f^\omega * g^\omega$, where $g^\omega(t) = \overline{f^\omega(-t)}$. By (1.11) and Proposition 2.2(ii), we conclude that $(\mathfrak{F} * k^\omega)|\mathbb{J} \in C_0(\mathbb{J}, X)$. Consider the open covering $\{V(\omega) : \omega \in K\}$. By compactness, there is a finite sub-covering $\{V(\omega_1), \dots, V(\omega_n)\}$ of K . One has $k = \sum_{i=1}^n k^{\omega_i} \in \mathcal{S}(\mathbb{R})$, $\text{supp } \widehat{k}$ is compact, $\widehat{k} \geq 1$ on K and $(\mathfrak{F} * k)|\mathbb{J} \in C_0(\mathbb{J}, X)$. By Lemma 1.1, there is $h \in \mathcal{S}(\mathbb{R})$ such that $\widehat{h} \cdot \widehat{k} = 1$ on K . Again by (2.3) and Proposition 2.2 (ii), it follows that $(\mathfrak{F} * \psi)|\mathbb{J} = ((\mathfrak{F} * k) * h * \psi)|\mathbb{J} \in C_0(\mathbb{J}, X)$. By Proposition 1.3 (i), $\mathfrak{F} * \psi \in C_0(\mathbb{R}, X)$.

(vi) As in part (iv) we conclude that $(\mathfrak{F} * \psi)|\mathbb{J}$ is uniformly continuous. It follows that $\mathfrak{F} * \psi \in C_0(\mathbb{R}, X)$ by (2.5), part (i) and Proposition 1.3 (i). \blacktriangleleft

Proposition 2.7. *Assume $F \in L^1_{loc}(\mathbb{J}, X)$ and $sp_{C_0(\mathbb{J}, X), \mathcal{D}}(F) = \emptyset$. If $(\mathfrak{F} * \psi)|_{\mathbb{J}}$ is uniformly continuous for some $\psi \in \mathcal{D}(\mathbb{R})$, then $\mathfrak{F} * \psi \in C_0(\mathbb{R}, X)$.*

Proof. Let $\omega \in \mathbb{R}$. There is $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\widehat{\varphi}(\omega) \neq 0$ and $(\mathfrak{F} * \varphi)|_{\mathbb{J}} \in C_0(\mathbb{J}, X)$. By Proposition 1.3 (i), we get $\mathfrak{F} * \varphi \in C_0(\mathbb{R}, X)$. By (1.11) and Proposition 2.1 (ii), we have $(\mathfrak{F} * \psi) * \varphi = (\mathfrak{F} * \varphi) * \psi \in C_0(\mathbb{R}, X)$. By (2.5), $sp_{C_0(\mathbb{R}, X), \mathcal{S}}(\mathfrak{F} * \psi) \subset sp_{C_0(\mathbb{R}, X), \mathcal{D}}(\mathfrak{F} * \psi) \subset sp_{C_0(\mathbb{R}, X), \mathcal{D}}(F) = \emptyset$. The result follows from Theorem 2.6 (i). \P

The following example shows that the assumption of uniform continuity is essential in Proposition 2.7.

Example 2.8. *If $F(t) = e^t$ for $t \in \mathbb{R}$, then $sp_{C_0(\mathbb{J}, X), \mathcal{D}}(F) = \emptyset$ but $(F * \psi)|_{\mathbb{J}}$ is unbounded for each $\psi \in \mathcal{D}(\mathbb{R})$ with $\int_{-\infty}^{\infty} e^{-s}\psi(s) ds \neq 0$.*

Proof. For any $\omega \in \mathbb{R}$, choose $a > 0$ such that $\cos \omega t$ does not change sign on $[0, a]$. Take $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi > 0$ on $(0, a)$ and $\text{supp } \varphi \subset [0, a]$. Let $f(t) = \varphi(t)$ for $t \geq 0$, $f(t) = -e^{2t}\varphi(-t)$ for $t < 0$. It follows that $f \in \mathcal{D}(\mathbb{R})$, $F * f = 0$ and $\widehat{f}(\omega) \neq 0$. This means $sp_{C_0(\mathbb{J}, X), \mathcal{D}}(F) = \emptyset$. Moreover, for $\psi \in \mathcal{D}(\mathbb{R})$ we have $F * \psi(t) = ce^t$, where $c = \int_{-\infty}^{\infty} e^{-s}\psi(s) ds$. So, $(F * \psi)|_{\mathbb{J}}$ is unbounded if $c \neq 0$. \P

In the following example we calculate the reduced spectra of some functions whose Fourier transforms may not be regular distributions.

Example 2.9. (i) *If $F \in L^p(\mathbb{J}, X)$ for some $1 \leq p < \infty$, then $M_h F \in C_0(\mathbb{J}, X)$ for all $h > 0$ and $sp_{C_0(\mathbb{R}, X), V}(F) = \emptyset$, where $V \in \{\mathcal{D}(\mathbb{R}), \mathcal{S}(\mathbb{R})\}$.*

(ii) *Let $F \in \mathcal{E}_{ub}(\mathbb{J}, X)$ and either $F' \in L^p(\mathbb{J}, X)$ for some $1 \leq p < \infty$ or more generally $F' \in L^1_{loc}(\mathbb{J}, X)$ with $M_h F' \in C_0(\mathbb{J}, X)$ for all $h > 0$. Then $F \in X \oplus C_0(\mathbb{J}, X)$ and $sp_{C_0(\mathbb{J}, X)}(F) \subset \{0\}$.*

Proof. (i) By Hölder's inequality, $\|M_h F(t)\| = (1/h) \|\int_0^h F(t+s) ds\| \leq h^{-1/p} (\int_0^h \|F(t+s)\|^p ds)^{1/p}$, so $M_h F \in C_0(\mathbb{J}, X)$ for all $h > 0$. By (1.7) and Proposition 1.3 (i), we get $\mathfrak{F} * s_h \in C_0(\mathbb{R}, X)$ for all $h > 0$. So, $sp_{C_0(\mathbb{R}, X), V}(\mathfrak{F} * s_h) = sp_{C_0(\mathbb{J}, X), V}(M_h F) = \emptyset$ for all $h > 0$. Hence $sp_{C_0(\mathbb{R}, X), V}(F) = \emptyset$ by (2.5).

(ii) By part (i) we have $hM_h F'(\cdot) = F(\cdot + h) - F(\cdot) \in C_0(\mathbb{J}, X)$ for all $h > 0$. Let $\tilde{F} \in BUC(\mathbb{R}, X)$ be given by $\tilde{F} = F$ on \mathbb{J} and $\tilde{F}(t) = F(0)$ on $\mathbb{R} \setminus \mathbb{J}$. It follows that $\Delta_s \tilde{F} \in C_0(\mathbb{R}, X)$ for all $s \in \mathbb{R}$. By [5, Theorems 4.2.2, Corollary 4.2.3], we conclude that $F = \tilde{F}|_{\mathbb{J}} \in X \oplus C_0(\mathbb{J}, X)$. This implies $sp_{C_0(\mathbb{J}, X)}(F) \subset \{0\}$. \P

The following result is due to Chill [17, Proposition 2.1]. The proof below is direct and shorter. It follows in particular that the assumptions of Proposition 2.7 are implied by the assumptions of Example 2.10.

Example 2.10. *If $F \in L^1_{loc}(\mathbb{R}, X)$ and if the Fourier transform \widehat{F} defined by (1.3) belongs to $\mathcal{S}'_{ar}(\mathbb{R}, X)$ and if $\psi \in \mathcal{D}(\mathbb{R})$, then $F * \psi \in C_0(\mathbb{R}, X)$ and so $sp_{C_0(\mathbb{R}, X), \mathcal{D}}(F) = \emptyset$.*

Proof. By (1.1), we have $G = 1/(2\pi)\widehat{F}\widehat{\psi} \in L^1(\mathbb{R}, X)$. By (1.3)

$$F * \psi(t) = \int_{-\infty}^{\infty} F(s)\psi(t-s) ds = \langle F, (\check{\psi})_{-t} \rangle = (1/2\pi) \int_{-\infty}^{\infty} \widehat{F}(\eta)e^{it\eta}\widehat{\psi}(\eta) d\eta.$$

This means that $F * \psi(t) = \widehat{G}(-t)$. By the Riemann-Lebesgue lemma, $F * \psi \in C_0(\mathbb{R}, X)$. This implies $sp_{C_0(\mathbb{R}, X), \mathcal{D}}(F) = \emptyset$. \P

§3. PROPERTIES OF THE WEAK LAPLACE SPECTRA

In this section we establish some new properties of the (weak) Laplace spectrum for regular tempered distributions and show that they are similar to those of the Carleman spectrum (see [26, Proposition 0.6]). We use the functions e_a for $a \geq 0$ defined on \mathbb{R} or \mathbb{R}_+ by $e_a(t) = e^{-at}$.

If $F \in \mathcal{S}'_{ar}(\mathbb{R}_+, X)$, then $e_a F \in L^1(\mathbb{R}_+, X)$ for all $a > 0$ and so the *Laplace transform* $\mathcal{L}F$ may be defined by

$$(3.1) \quad \mathcal{L}F(\lambda) = \int_0^{\infty} e^{-\lambda t} F(t) dt \quad \text{for } \lambda \in \mathbb{C}_+.$$

For a function $F \in \mathcal{S}'_{ar}(\mathbb{R}, X)$ the *Carleman transform* $\mathcal{C}F$ is defined by

$$(3.2) \quad \mathcal{C}F(\lambda) = \begin{cases} \mathcal{L}^+ F(\lambda) = \int_0^{\infty} e^{-\lambda t} F(t) dt & \text{for } \lambda \in \mathbb{C}_+ \\ \mathcal{L}^- F(\lambda) = - \int_0^{\infty} e^{\lambda t} F(-t) dt & \text{for } \lambda \in \mathbb{C}_-. \end{cases}$$

If $F \in L^1(\mathbb{R}_+, X)$, then $\mathcal{L}F$ has a continuous extension to $\mathbb{C}_+ \cup i\mathbb{R}$ given also by the integral in (3.1). By the Riemann-Lebesgue lemma $\widehat{\mathfrak{F}} = \mathcal{L}F(i \cdot) \in C_0(\mathbb{R}, X)$.

If $F \in \mathcal{S}'_{ar}(\mathbb{R}_+, X)$, then $\widehat{\mathfrak{F}} \in \mathcal{S}'(\mathbb{R}, X)$ and $\mathcal{L}F(a + i \cdot) = \widehat{e_a \mathfrak{F}} \in \mathcal{S}'_{ar}(\mathbb{R}, X)$ for all $a > 0$. Moreover, for $\varphi \in \mathcal{S}(\mathbb{R})$,

$$(3.3) \quad \langle \mathcal{L}F(a + i \cdot), \varphi \rangle = \langle \widehat{e_a \mathfrak{F}}, \varphi \rangle = \langle e_a \mathfrak{F}, \widehat{\varphi} \rangle \rightarrow \langle \mathfrak{F}, \widehat{\varphi} \rangle = \langle \widehat{\mathfrak{F}}, \varphi \rangle,$$

where the limit exists as $a \searrow 0$ by the Lebesgue convergence theorem. This means that $\lim_{a \searrow 0} \mathcal{L}F(a + i \cdot) = \widehat{\mathfrak{F}}$ with respect to the weak dual topology on $\mathcal{S}'(\mathbb{R}, X)$.

For a holomorphic function $\zeta : \Sigma \rightarrow X$, where $\Sigma = \mathbb{C}_+$ or $\Sigma = \mathbb{C} \setminus i\mathbb{R}$, the point $i\omega \in i\mathbb{R}$ is called a *regular point* for ζ or ζ is called *holomorphic* at $i\omega$, if ζ has an extension $\overline{\zeta}$ which is holomorphic in a neighbourhood $V \subset \mathbb{C}$ of $i\omega$.

Points $i\omega$ which are *not regular points* are called *singular points*.

The *Laplace spectrum* of a function $F \in \mathcal{S}'_{ar}(\mathbb{R}_+, X)$ is defined by

$$(3.4) \quad sp^{\mathcal{L}}(F) = \{\omega \in \mathbb{R} : i\omega \text{ is a singular point for } \mathcal{L}F\}.$$

The *Carleman spectrum* of a function $F \in \mathcal{S}'_{ar}(\mathbb{R}, X)$ is defined by

$$(3.5) \quad sp^{\mathcal{C}}(F) = \{\omega \in \mathbb{R} : i\omega \text{ is a singular point for } \mathcal{C}F\}. \text{ See [4, (4.26)]}.$$

The Laplace spectrum is also called the half-line spectrum ([4, p. 275]).

Note that if $\overline{\mathcal{L}}\gamma_{-\omega}F$ and $\overline{\mathcal{C}}\gamma_{-\omega}F$ are holomorphic extensions of $\mathcal{L}\gamma_{-\omega}F$ and $\mathcal{C}\gamma_{-\omega}F$ respectively, which are holomorphic in a neighbourhood of 0, then

$$(3.6) \quad \begin{aligned} \lim_{\lambda \rightarrow 0} \mathcal{L}\gamma_{-\omega}F(\lambda) &= \overline{\mathcal{L}}\gamma_{-\omega}F(0) \text{ if } \omega \notin sp^{\mathcal{L}}(F), \text{ and} \\ \lim_{\lambda \rightarrow 0} \mathcal{C}\gamma_{-\omega}F(\lambda) &= \overline{\mathcal{C}}\gamma_{-\omega}F(0) \text{ if } \omega \notin sp^{\mathcal{C}}(F). \end{aligned}$$

If $F \in L^\infty(\mathbb{R}_+, X)$ and $sp^{\mathcal{L}}(F) = \emptyset$, then by Zagier's result [31, Analytic Theorem] we conclude that $\widehat{F}(\omega) = \int_0^\infty e^{-i\omega t} F(t) dt$ exists as an improper integral (and by (3.6) equals $\overline{\mathcal{L}}\gamma_{-\omega}F(0)$) for each $\omega \in \mathbb{R}$. Zagier's Analytic Theorem does not hold for unbounded functions. Indeed, the Laplace spectrum of $F(t) = te^{it^2}$ is empty (see Example 3.4 below) and it can be verified that $\int_0^\infty e^{-i\omega t} F(t) dt$ does not exist as an improper Riemann integral for any $\omega \in \mathbb{R}$.

For a holomorphic function $\zeta : \mathbb{C}_+ \rightarrow X$, the point $i\omega \in i\mathbb{R}$ is called a *weakly regular point* for ζ if there exist $\varepsilon > 0$ and $h \in L^1(\omega - \varepsilon, \omega + \varepsilon)$ such that

$$(3.7) \quad \begin{aligned} \lim_{a \searrow 0} \int_{-\infty}^\infty \zeta(a + is) \varphi(s) ds &= \int_{\omega - \varepsilon}^{\omega + \varepsilon} h(s) \varphi(s) ds \\ \text{for all } \varphi \in \mathcal{D}(\mathbb{R}) \text{ with } \text{supp } \varphi &\subset (\omega - \varepsilon, \omega + \varepsilon). \end{aligned}$$

Points $i\omega$ which are not weakly regular points are called *weakly singular points*.

The *weak Laplace spectrum* of $F \in \mathcal{S}'_{ar}(\mathbb{R}_+, X)$ is defined by ([4, p. 324])

$$(3.8) \quad sp^{w\mathcal{L}}(F) = \{\omega \in \mathbb{R} : i\omega \text{ is not a weakly regular point for } \mathcal{L}F\}.$$

For $F \in \mathcal{S}'_{ar}(\mathbb{R}, X)$, we write $sp^{w\mathcal{L}^+}(F) = sp^{w\mathcal{L}}(F|_{\mathbb{R}_+})$. It follows readily that if $F \in \mathcal{S}'_{ar}(\mathbb{R}, X)$ then

$$(3.9) \quad sp^{w\mathcal{L}^+}(F) \subset sp^{\mathcal{L}^+}(F) \subset sp^{\mathcal{C}}(F); \text{ and, if } F \in L^1(\mathbb{R}_+, X), sp^{w\mathcal{L}}(F) = \emptyset.$$

In the following sp^* denotes $sp^{\mathcal{L}^+}$ or $sp^{w\mathcal{L}^+}$ or $sp^{\mathcal{C}}$.

Proposition 3.1. *If $F \in \mathcal{S}'_{ar}(\mathbb{R}, X)$, then*

- (i) $sp^*(F) = sp^*(F_s)$ for each $s \in \mathbb{R}$.
- (ii) $sp^*(F) = \cup_{h>0} sp^*(M_h F)$.
- (iii) $sp^*(\gamma_\omega F) = \omega + sp^*(F)$.

Proof. (i) A simple calculation shows that for $\lambda \in \mathbb{C}^\pm$

$$(3.10) \quad \mathcal{L}^\pm F_s(\lambda) = e^{\lambda s} \mathcal{L}^\pm F(\lambda) - e^{\lambda s} \int_0^s e^{-\lambda t} F(t) dt.$$

Note that the second term on the right of (3.10) is entire in λ for each $s \in \mathbb{R}$. It follows that \mathcal{L}^+F (respectively $\mathcal{C}F$) is holomorphic at $i\omega$ if and only if \mathcal{L}^+F_s (respectively $\mathcal{C}F_s$) is holomorphic at $i\omega$. This proves (i) for $sp^{\mathcal{L}^+}$ and $sp^{\mathcal{C}}$. Now, assume $i\omega$ is a weakly regular point for \mathcal{L}^+F . So there exists $\varepsilon > 0$ and $h \in L^1(\omega - \varepsilon, \omega + \varepsilon)$ satisfying $\lim_{a \searrow 0} \int_{-\infty}^{\infty} \mathcal{L}F(a + i\eta)\varphi(\eta) d\eta = \int_{\omega-\varepsilon}^{\omega+\varepsilon} h(\eta)\varphi(\eta) d\eta$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ with $\text{supp } \varphi \subset (\omega - \varepsilon, \omega + \varepsilon)$. Then by [28, Theorem 6.18, p. 146] (valid also for X -valued distributions), $\lim_{a \searrow 0} \int_{-\infty}^{\infty} \mathcal{L}^+F(a + i\eta)e^{(a+i\eta)s}\varphi(\eta)d\eta = \int_{\omega-\varepsilon}^{\omega+\varepsilon} h(\eta)e^{i\eta s}\varphi(\eta)d\eta$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ with $\text{supp } \varphi \subset (\omega - \varepsilon, \omega + \varepsilon)$. It follows that $i\omega$ is a weakly regular point for \mathcal{L}^+F_s .

(ii) Another calculation shows that for $\lambda \in \mathbb{C}^\pm$

$$(3.11) \quad \mathcal{L}^\pm M_h F(\lambda) = g(\lambda h) \mathcal{L}^\pm F(\lambda) - (1/h) \int_0^h (e^{\lambda v} \int_0^v e^{-\lambda t} F(t) dt) dv,$$

where g is the entire function given by $g(\lambda) = \frac{e^\lambda - 1}{\lambda}$ for $\lambda \neq 0$. Let $i\omega \in i\mathbb{R}$ be a regular point for \mathcal{L}^+F and let $\overline{\mathcal{L}^+}F : V \rightarrow X$ be a holomorphic extension of \mathcal{L}^+F to a neighbourhood $V \subset \mathbb{C}$ of $i\omega$. Then $\overline{\mathcal{L}^+}M_h F(\lambda) = g(\lambda h) \overline{\mathcal{L}^+}F(\lambda) - (1/h) \int_0^h (e^{\lambda v} \int_0^v e^{-\lambda t} F(t) dt) dv$, $\lambda \in V$, is a holomorphic extension of $\mathcal{L}^+M_h F$. So $i\omega$ is a regular point for $\mathcal{L}^+M_h F$. Conversely suppose $i\omega \in i\mathbb{R}$ is a regular point of $\mathcal{L}^+M_h F$ for each $h > 0$. Choose $h_0 > 0$ such that $g(i\omega h_0) \neq 0$. Then $i\omega$ is a regular point for \mathcal{L}^+F . This proves (ii) for $sp^{\mathcal{L}^+}$. The case $sp^{\mathcal{C}}$ follows similarly noting that (3.11) implies $\mathcal{C}M_h F(\lambda) = g(\lambda h) \mathcal{C}F(\lambda) - (1/h) \int_0^h (e^{\lambda v} \int_0^v e^{-\lambda t} F(t) dt) dv$. The proof for $sp^{w\mathcal{L}^+}$ is similar to the one in part (i).

(iii) This follows easily from the definitions noting that $\mathcal{L}^+(\gamma_\omega F)(\lambda) = \mathcal{L}^+F(\lambda - i\omega)$ and $\mathcal{C}(\gamma_\omega F)(\lambda) = \mathcal{C}F(\lambda - i\omega)$. \P

Proposition 3.1 holds for \mathfrak{F} , where $F \in \mathcal{S}'_{ar}(\mathbb{R}_+, X)$. In this case $sp^{\mathcal{L}^+}\mathfrak{F} = sp^{\mathcal{C}}F$ and $sp^{w\mathcal{L}^+}\mathfrak{F} = sp^{w\mathcal{L}}F$.

The following result was obtained in [16, Lemma 1.16] in the case $\mathcal{A} = C_0(\mathbb{R}_+, X)$ and $F \in L^\infty(\mathbb{R}_+, X)$ since then $sp_{C_0(\mathbb{R}_+, X), \mathcal{S}}(F) = sp_{C_0(\mathbb{R}_+, X)}(F)$.

Proposition 3.2. *If $F \in \mathcal{S}'_{ar}(\mathbb{R}_+, X)$ and $\mathcal{A} \subset L^\infty(\mathbb{R}_+, X)$ satisfies (2.1) then $sp_{\mathcal{A}, \mathcal{S}}(F) \subset sp_{C_0(\mathbb{R}_+, X), \mathcal{S}}(F) \subset sp^{w\mathcal{L}}(F)$.*

Proof. By Proposition 2.1 (i), $C_0(\mathbb{R}_+, X) \subset \mathcal{A}$ and so $sp_{\mathcal{A}, \mathcal{S}}(F) \subset sp_{C_0(\mathbb{R}_+, X), \mathcal{S}}(F)$. Let $\omega \notin sp^{w\mathcal{L}}(F)$. Choose $\varepsilon > 0$ and $\varphi \in \mathcal{S}(\mathbb{R})$ such that $sp^{w\mathcal{L}}(F) \cap [\omega - \varepsilon, \omega + \varepsilon] = \emptyset$, $\widehat{\varphi}(\omega) = 1$ and $\text{supp } \widehat{\varphi} \subset [\omega - \varepsilon, \omega + \varepsilon]$. By [17, Proposition 1.3], $\mathfrak{F} * \varphi \in C_0(\mathbb{R}, X)$ and so $\omega \notin sp_{C_0(\mathbb{R}_+, X), \mathcal{S}}(F)$. \P

Remark 3.3. (i) In the case $\mathbb{J} = \mathbb{R}_+$ Theorem 2.5 and Theorem 2.6 remain valid if we replace $sp_{\mathcal{A}, \mathcal{S}}(F)$ and $sp_{C_0(\mathbb{R}_+, X), \mathcal{S}}(F)$ by $sp^{\mathcal{L}}(F)$ or $sp^{w\mathcal{L}}(F)$. Indeed, note that Theorem 2.5 holds for $\mathcal{A} = C_0(\mathbb{R}_+, X)$. By Proposition 3.2 and (3.9), we have $sp_{C_0(\mathbb{R}_+, X), \mathcal{S}}(F) \subset sp^{w\mathcal{L}}(F) \subset sp^{\mathcal{L}}(F)$.

(ii) If F in Theorem 2.5 is not bounded or slowly oscillating, then F is not necessarily ergodic. For example, if $\mathbf{g}(t) = e^{it^2}$ and $F = \mathbf{g}^{(n)}$ for some $n \in \mathbb{N}$, then by Example 3.4 below and (3.9), we find $sp^{w\mathcal{L}^+}(F) = \emptyset$. By Proposition 3.2, we get $sp_{C_0(\mathbb{R}_+, \mathbb{C})}(F) = \emptyset$ but $F|_{\mathbb{R}_+}$ is neither bounded nor ergodic when $n \geq 2$. If $n = 1$, F is ergodic but not bounded.

(iii) In view of Proposition 3.2 and (3.9) several tauberian theorems by Ingham [21] ([4, Theorem 4.9.5]) and their generalizations in [2], [3], [4, Theorem 4.7.7, Corollary 4.7.10, Theorem 4.9.7, Lemma 4.10.2], [13], [14], [16, Lemma 1.16, p. 25], [17] are consequences of Theorem 2.5 and Theorem 2.6. Our proofs are simpler and different. Replacing Laplace and weak Laplace spectra by reduced spectra we are able to strengthen and unify these previous results.

In the following we use our results to calculate some (weak) Laplace spectra.

Example 3.4. Take $\mathbf{g}(t) = e^{it^2}$ for $t \in \mathbb{R}$. Then $sp^{\mathcal{C}}(\mathbf{g}) = \mathbb{R}$ and $sp^{\mathcal{L}^+}(\mathbf{g}) = sp^{\mathcal{L}^+}(\mathbf{g}^{(n)}) = \emptyset$ for any $n \in \mathbb{N}$. Moreover, $M_h \mathbf{g} \in C_0(\mathbb{R}, \mathbb{C})$ and $sp^{\mathcal{L}^+}(M_h \mathbf{g}) = \emptyset$ for all $h > 0$.

Proof. By Proposition 3.1 (i), (iii), it is readily verified that $sp^{\mathcal{L}^+}(\mathbf{g}) = sp^{\mathcal{L}^+}(\mathbf{g}_a) = 2a + sp^{\mathcal{L}^+}(\mathbf{g})$ for each $a \in \mathbb{R}$. This implies that either $sp^{\mathcal{L}^+}(\mathbf{g}) = \emptyset$ or $sp^{\mathcal{L}^+}(\mathbf{g}) = \mathbb{R}$. Similarly either $sp^{\mathcal{C}}(\mathbf{g}) = \emptyset$ or $sp^{\mathcal{C}}(\mathbf{g}) = \mathbb{R}$. But $\mathbf{g} \neq 0$ and so by [26, Proposition 0.5 (ii)], $sp^{\mathcal{C}}(\mathbf{g}) = \mathbb{R}$.

Next note that $y(\lambda) = \mathcal{L}^+ \mathbf{g}(\lambda)$ is a solution of the differential equation $y'(\lambda) + (\lambda/2i)y(\lambda) = 1/2i$ for $\lambda \in \mathbb{C}_+$. Solving the equation we find $y(\lambda) = e^{-\lambda^2/4i}(c + (1/2i) \int_0^\lambda e^{z^2/4i} dz)$ for some choice of $c \in \mathbb{C}$. As this last function is entire we conclude that $sp^{\mathcal{L}^+}(\mathbf{g}) = \emptyset$. Since $\int_0^\infty e^{it^2} dt$ converges as an improper Riemann integral and $M_h \mathbf{g}(t) = P \mathbf{g}(t+h) - P \mathbf{g}(t)$ it follows that $M_h \mathbf{g} \in C_0(\mathbb{R}, \mathbb{C})$ for each $h > 0$. Moreover, by Proposition 3.1(ii), $sp^{\mathcal{L}^+}(M_h \mathbf{g}) = \emptyset$. \P

Finally, we demonstrate that our results can be used to deduce spectral criteria for bounded solutions of evolution equations of the form

$$(3.12) \quad \frac{du(t)}{dt} = Au(t) + \phi(t), \quad u(0) \in X, \quad t \in \mathbb{J},$$

where A is a closed linear operator on X and $\phi \in L^\infty(\mathbb{J}, X)$.

Theorem 3.5. *Let $\phi \in L^\infty(\mathbb{J}, X)$ and u be a bounded mild solution of (3.12). Let \mathcal{A} satisfy (2.1), (2.3), $\gamma_\lambda \mathcal{A} \subset \mathcal{A}$ for all $\lambda \in \mathbb{R}$ and contain all constants.*

- (i) *If $\mathbb{J} = \mathbb{R}_+$, then $i \operatorname{sp}^\mathcal{L}(u) \subset (\sigma(A) \cap i\mathbb{R}) \cup i \operatorname{sp}^\mathcal{L}(\phi)$.*
- (ii) *If $\operatorname{sp}_\mathcal{A}(\phi) = \emptyset$, then $i \operatorname{sp}_\mathcal{A}(u) \subset \sigma(A) \cap i\mathbb{R}$.*

Proof. As $u, \phi \in L^\infty(\mathbb{J}, X)$ we get $M_h u, M_h \phi \in BUC(\mathbb{J}, X)$ and $v = M_h u$ is a classical solution of $v'(t) = Av(t) + M_h \phi(t)$, $v(0) \in D(A)$, $t \in \mathbb{J}$ for each $h > 0$.

(i) By [4, Proposition 5.6.7, p. 380], we have

$$i \operatorname{sp}^\mathcal{L}(M_h u) \subset (\sigma(A) \cap i\mathbb{R}) \cup i \operatorname{sp}^\mathcal{L}(M_h \phi) \text{ for all } h > 0.$$

Taking the union of both sides, we get

$$\cup_{h>0} i \operatorname{sp}^\mathcal{L}(M_h u) \subset (\sigma(A) \cap i\mathbb{R}) \cup (\cup_{h>0} i \operatorname{sp}^\mathcal{L}(M_h \phi)).$$

Applying Proposition 3.1 (ii) to both sides, we conclude that

$$i \operatorname{sp}^\mathcal{L}(u) \subset (\sigma(A) \cap i\mathbb{R}) \cup i \operatorname{sp}^\mathcal{L}(\phi).$$

(ii) Take $h > 0$. Since $\operatorname{sp}_\mathcal{A}(\phi) = \emptyset$, it follows that $\operatorname{sp}_\mathcal{A}(M_h \phi) = \emptyset$, by (2.5). Hence $M_h \phi \in \mathcal{A}$ by [5, Theorem 4.2.1]. Using [6, Corollary 3.4 (i)], we conclude that $i \operatorname{sp}_\mathcal{A}(M_h u) \subset \sigma(A) \cap i\mathbb{R}$. Again by (2.5), we conclude that $i \operatorname{sp}_\mathcal{A}(u) \subset \sigma(A) \cap i\mathbb{R}$. For further details see [10, Proposition 4.2, Theorem 4.3]. ◻

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